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Quantum Dynamics of a Dissipative Deformed Harmonic Oscillator

M. Daeimohamad · F. Kheirandish · M.R. Abolhassani

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Abstract The quantum dynamics of a dissipative deformed harmonic oscillator is investigated in the framework of the minimal coupling method. The reduced density matrix of the deformed oscillator is obtained and the decay transitions are calculated.

Keywords Deformed · Dissipative · Minimal coupling method

1 Introduction

For more than a decade a constant interest has been induced to the study of deformations of Lie algebras. The rich structure of these algebras has produced many important results and consequences in quantum and conformal field theories, statistical mechanics, quantum and nonlinear optics, nuclear and molecular physics. The applications of these algebras in physics became intense with the introduction in 1989 by Biedenharn [1] and MacFarlane [2], of q-deformed Weyl-Heisenberg algebra, that is deformed quantum harmonic oscillator. Since then many properties and generalizations of deformed oscillators have been investigated. The deformed oscillators are important since they are the main building blocks of integrable models. Also they are closely related to nonlinearity. It has been shown that the deformed oscillators lead to nonlinear vibrations with a special kind of the dependence of the frequency on the amplitude [3, 4]. A correction to the Plank distribution formula produced by a deformed Bose distribution is also reported in [3, 5, 6].

In the present paper we intend to study the quantum dynamics of a dissipative deformed harmonic oscillator in the framework of the minimal coupling method [7–9].

M. Daeimohamad · M.R. Abolhassani
Department of Physics, Science and Research Campus, Azad University of Tehran, Tehran, Iran

M. Daeimohamad
e-mail: daeimohamad@phys.ui.ac.ir

F. Kheirandish (✉)
Department of Physics, University of Isfahan, Hezar Jarib Ave., Isfahan, Iran
e-mail: fardin_kh@phys.ui.ac.ir

2 Quantum Dynamics

A non deformed harmonic oscillator is described by the Weyl-Heisenberg algebra

$$\begin{aligned}
[\hat{N}, \hat{a}] &= -\hat{a}, \\
[\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger, \\
[\hat{a}, \hat{a}^\dagger] &= 1,
\end{aligned} \tag{1}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator and the Hamiltonian is defined by $\hat{H} = \hbar\omega_0(\hat{N} + \frac{1}{2})$.

For a deformation of the above algebra let us introduce the deformed operators $\hat{A}$, $\hat{A}^\dagger$ as follows [3]

$$\begin{aligned}
\hat{A} &= \hat{a} f(\hat{N}) = f(\hat{N}+1)\hat{a}, \\
\hat{A}^\dagger &= f(\hat{N})\hat{a}^\dagger = \hat{a}^\dagger f(\hat{N}+1),
\end{aligned} \tag{2}$$

where $f(\hat{N})$ is the deformation operator and for $f(\hat{N}) \equiv 1$, we recover the non deformed algebra. The deformed operators satisfy the following deformed algebra

$$\begin{aligned}
[\hat{N}, \hat{A}] &= -\hat{A}, \\
[\hat{N}, \hat{A}^\dagger] &= \hat{A}^\dagger, \\
[\hat{A}, \hat{A}^\dagger] &= (\hat{N}+1) f^2(\hat{N}+1) - \hat{N} f^2(\hat{N}).
\end{aligned} \tag{3}$$

In order to apply the minimal coupling method to the Hamiltonian of the deformed oscillator we need to find the deformed position and momentum operators. This can be achieved by replacing $\hat{a}$ and $\hat{a}^\dagger$ with $\hat{A}$ and $\hat{A}^\dagger$ in the usual definitions of $\hat{q}$ and $\hat{p}$ as

$$\begin{aligned}
\hat{q} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{A} + \hat{A}^\dagger), \\
\hat{p} &= \imath\sqrt{\frac{m\hbar\omega}{2}}(\hat{A}^\dagger - \hat{A}).
\end{aligned} \tag{4}$$

The Hamiltonian of the deformed oscillator is now defined by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2. \tag{5}$$

For investigating the dissipative quantum dynamics of a deformed harmonic oscillator we introduce a reservoir defined by the field $\hat{R}$ and couple it minimally to $\hat{p}$ as [7, 8]

$$\begin{aligned}
\hat{H} &= \frac{(\hat{p} - \hat{R})^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2 + \hat{H}_B \\
&= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2 + \frac{\hat{R}^2}{2m} - \frac{\hat{R}\hat{p}}{m} + \hat{H}_B,
\end{aligned} \tag{6}$$

where the field $\hat{R}$ is defined by

$$\hat{R}(t) = \int_{-\infty}^{\infty} dk [f_1(\omega_k)\hat{b}_k(t) + f_1^*(\omega_k)\hat{b}_k^\dagger(t)], \tag{7}$$

with the following Hamiltonian

$$\hat{H}_B = \int_{-\infty}^{\infty} dk\hbar\omega_k \hat{b}_k^\dagger \hat{b}_k, \tag{8}$$

as the Hamiltonian of the reservoir. The frequency dependent coupling function $f_1(\omega_k)$, plays the basic role in the interaction between the deformed oscillator and each mode of the reservoir. For finding the time-evolution of the total system we work in the interaction picture and so decompose the Hamiltonian as

$$\begin{aligned} \hat{H}_0 &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + \hat{H}_B, \\ \hat{H}' &= -\frac{\hat{R}\hat{p}}{m} + \frac{\hat{R}^2}{2m}. \end{aligned} \tag{9}$$

For convenience we assume that the deformed oscillator is coupled to the reservoir weakly such that we can ignore the quadratic term $\frac{\hat{R}^2}{2m}$ in (9). So, from now on we assume

$$\hat{H}' = -\frac{\hat{R}\hat{p}}{m}. \tag{10}$$

In interaction picture we have

$$\begin{aligned} \hat{H}_I'(t) &= e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}'(0)e^{-\frac{i}{\hbar}\hat{H}_0 t} \\ &= -\frac{1}{m}\hat{R}_I\hat{p}_I, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \hat{R}_I &= e^{\frac{i}{\hbar}\hat{H}_B t}\hat{R}(0)e^{-\frac{i}{\hbar}\hat{H}_B t} \\ &= \int_{-\infty}^{\infty} dk[f_1(\omega_k)e^{-\imath\omega_k t}\hat{b}_k(0) + f_1^*(\omega_k)e^{\imath\omega_k t}\hat{b}_k^\dagger(0)], \end{aligned} \tag{12}$$

and

$$\begin{aligned} \hat{p}_I &= \imath\sqrt{\frac{m\hbar\omega}{2}}(\hat{A}_I^\dagger - \hat{A}_I), \\ \hat{A}_I(t) &= e^{\frac{\imath\omega t}{2}(\hat{A}^\dagger\hat{A}+\hat{A}\hat{A}^\dagger)}\hat{A}(0)e^{-\frac{\imath\omega t}{2}(\hat{A}^\dagger\hat{A}+\hat{A}\hat{A}^\dagger)}. \end{aligned} \tag{13}$$

Let us introduce the new operator $\gamma(\hat{N})$ by

$$\hat{A}^\dagger\hat{A} + \hat{A}\hat{A}^\dagger = \gamma(\hat{N}), \tag{14}$$

the usual eigenkets $|n\rangle$ of the non deformed oscillator are also the eigenkets of the deformed oscillator as it can easily be checked. The Hamiltonian (5) in terms of $\gamma(\hat{N})$ is written as $\hat{H} = \frac{\hbar\omega}{2}\gamma(\hat{N})$, with eigenvalues $E_d = \frac{\hbar\omega}{2}\gamma(n)$. In this basis the matrix elements of $\hat{A}$ and $\hat{A}^\dagger$ are given by

$$\begin{aligned} \hat{A}^\dagger_{n,m} &= \langle n|f(\hat{N})\hat{a}^\dagger|m\rangle = \sqrt{m+1}f(m+1)\delta_{n,m+1}, \\ \hat{A}_{n,m} &= \langle n|\hat{a}f(\hat{N})|m\rangle = \sqrt{m}f(m)\delta_{n,m-1}, \end{aligned} \tag{15}$$

using the relations (13) and (15) we have

$$p_{I;n,m} = \langle n|\hat{p}_I(t)|m\rangle = \imath\sqrt{\frac{m\hbar\omega}{2}}e^{\frac{\imath\omega t}{2}(\gamma(n)-\gamma(m))}(\hat{A}^\dagger_{nm}(0) - \hat{A}_{nm}(0)),$$
$$= \imath\sqrt{\frac{m\hbar\omega}{2}}e^{\frac{\imath\omega t}{2}(\gamma(n)-\gamma(m))}$$
$$\times (\sqrt{m+1}f(m+1)\delta_{n,m+1} - \sqrt{m}f(m)\delta_{n,m-1}). \tag{16}$$

Therefore the interaction term $\hat{H}'$, can be written in interaction picture as

$$\hat{H}'_I = -\frac{1}{m}\hat{p}_I\hat{R}_I$$
$$= -\frac{1}{m}\int_{-\infty}^{+\infty} dk[f_1(\omega_k)e^{-\imath\omega_k t}\hat{p}_I(t)\hat{b}_k(0) + f_1^*(\omega_k)e^{\imath\omega_k t}\hat{p}_I(t)\hat{b}_k^\dagger(0)]. \tag{17}$$

The time-evolution operator $\hat{U}_I$, up to the first-order time-dependent perturbation is

$$\hat{U}_I(t,0) = 1 - \frac{\imath}{\hbar}\int_0^t dt_1\hat{H}'_I(t_1)$$
$$= 1 + \frac{\imath}{m\hbar}\int_0^t\int_{-\infty}^{\infty} dk\,\hat{p}_I(t_1)[f_1(\omega_k)e^{-\imath\omega_k t_1}\hat{b}_k(0) + f_1^*(\omega_k)e^{\imath\omega_k t_1}\hat{b}_k^\dagger(0)]dt_1. \tag{18}$$

Having the time-evolution operator, we can find the density operator $\hat{\rho}_I(t)$ of the total system in any time t, as follows

$$\hat{\rho}_I(t) = \hat{U}_I(t)\hat{\rho}_I(0)\hat{U}_I^\dagger(t) = \hat{\rho}_I(0)$$
$$+ \frac{\imath}{m\hbar}\int_0^t dt_1\int_{-\infty}^{\infty} dk\,\hat{p}_I(t_1)\{f_1(\omega_k)e^{-\imath\omega_k t_1}\hat{b}_k + f_1^*(\omega_k)e^{\imath\omega_k t_1}\hat{b}_k^\dagger\}\hat{\rho}_I(0)$$
$$- \frac{\imath}{m\hbar}\int_0^t dt_1\int_{-\infty}^{\infty} dk\,\hat{\rho}_I(0)\{f_1^*(\omega_k)e^{\imath\omega_k t_1}\hat{b}_k^\dagger + f_1(\omega_k)e^{-\imath\omega_k t_1}\hat{b}_k\}\hat{p}_I(t_1)$$
$$+ \frac{1}{(m\hbar)^2}\int_0^t dt_1\int_{-\infty}^{\infty} dk\int_0^t dt_2\int_{-\infty}^{\infty} dk'$$
$$\times \{f_1(\omega_k)e^{-\imath\omega_k t_1}(\hat{p}_I(t_1)\hat{b}_k\hat{\rho}_I\hat{p}_I(t_2)\hat{b}_{k'}^\dagger)f_1^*(\omega_{k'})e^{\imath\omega_{k'}t_2}$$
$$+ f_1(\omega_k)e^{-\imath\omega_k t_1}(\hat{p}_I(t_1)\hat{b}_k\hat{\rho}_I\hat{p}_I(t_2)\hat{b}_{k'})f_1(\omega_{k'})e^{-\imath\omega_{k'}t_2}$$
$$+ f_1^*(\omega_k)e^{\imath\omega_k t_1}(\hat{p}_I(t_1)\hat{b}_k^\dagger\hat{\rho}_I\hat{p}_I(t_2)\hat{b}_{k'}^\dagger)f_1^*(\omega_{k'})e^{\imath\omega_{k'}t_2}$$
$$+ f_1^*(\omega_k)e^{\imath\omega_k t_1}(\hat{p}_I(t_1)\hat{b}_k^\dagger\hat{\rho}_I\hat{p}_I(t_2)\hat{b}_{k'})f_1(\omega_{k'})e^{-\imath\omega_{k'}t_2}. \tag{19}$$

Let us assume that the reservoir has a Maxwell-Boltzman thermal distribution given by

$$\hat{\rho}_B(0) = \frac{e^{-\frac{\hat{H}_B}{KT}}}{\mathrm{Tr}_B[e^{-\frac{\hat{H}_B}{KT}}]}, \tag{20}$$

where Tr_B, means taking trace over the degrees of freedom of the reservoir. Then the initial density matrix of the total system can be written as

$$\hat{\rho}_I(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0), \tag{21}$$

where $\hat{\rho}_S(0)$ is the initial density matrix of the deformed oscillator. For finding the reduced density matrix of the oscillator, we use the following relations which can be easily obtained

$$\begin{aligned} &\mathrm{Tr}_B[\hat{b}_k \hat{\rho}_B(0) \hat{b}_{k'}] = \mathrm{Tr}_B[\hat{b}_k^\dagger \hat{\rho}_B(0) \hat{b}_{k'}^\dagger] = 0, \\ &\mathrm{Tr}_B[\hat{b}_k \hat{\rho}_B(0) \hat{b}_{k'}^\dagger] = \frac{\delta(k-k')}{e^{\frac{\hbar\omega_k}{KT}} - 1}, \\ &\mathrm{Tr}_B[\hat{b}_k^\dagger \hat{\rho}_B(0) \hat{b}_{k'}] = \frac{\delta(k-k') e^{\frac{\hbar\omega_k}{KT}}}{e^{\frac{\hbar\omega_k}{KT}} - 1}. \end{aligned} \tag{22}$$

Now the reduced density matrix can be obtained by tracing out the reservoir degrees of freedom as

$$\begin{aligned} \hat{\rho}_S(t) &= \mathrm{Tr}_B[\hat{\rho}_I(t)], \\ &= \hat{\rho}_S(0) + \frac{1}{(m\hbar)^2} \int_0^t dt_1 \int_0^t dt_2 \int_{-\infty}^{\infty} dk \\ &\quad \times \left\{ \frac{|f_1(\omega_k)|^2 e^{\imath\omega_k(t_2-t_1)}}{e^{\beta\hbar\omega_k} - 1} \hat{p}_I(t_1) \hat{\rho}_S(0) \hat{p}_I(t_2) \right. \\ &\quad \left. + \frac{|f_1(\omega_k)|^2 e^{\beta\hbar\omega_k - \imath\omega_k(t_2-t_1)}}{e^{\beta\hbar\omega_k} - 1} \hat{p}_I(t_1) \hat{\rho}_S(0) \hat{p}_I(t_2) \right\}. \end{aligned} \tag{23}$$

Now as an example assume that the deformed oscillator is initially in its Nth excited state $|N\rangle$, that is the density matrix $\hat{\rho}_S(0)$ is a pure state initially

$$\hat{\rho}_S(0) = |N\rangle\langle N|. \tag{24}$$

Then the density matrix in an arbitrary time is

$$\begin{aligned} \hat{\rho}_{S;n,m} &= \delta_{n,N}\delta_{N,m} + \frac{1}{(m\hbar)^2} \int_0^t dt_1 \int_0^t dt_2 \int_{-\infty}^{\infty} dk \\ &\quad \times \left\{ \frac{|f_1(\omega_k)|^2 e^{\imath\omega_k(t_2-t_1)}}{e^{\beta\hbar\omega_k} - 1} + \frac{|f_1(\omega_k)|^2 e^{\beta\hbar\omega_k - \imath\omega_k(t_2-t_1)}}{e^{\beta\hbar\omega_k} - 1} \right\} \\ &\quad \times \langle n|\hat{p}_I(t_1)|N\rangle\langle N|\hat{p}_I(t_2)|m\rangle. \end{aligned} \tag{25}$$

The last term in (25) can be calculated explicitly as

$$
\begin{aligned}
&\langle n|\hat{p}_I(t_1)|N\rangle\langle N|\hat{p}_I(t_2)|m\rangle \\
&= -\frac{m\hbar\omega}{2}\begin{cases}
e^{\imath\omega\Omega(N-1)t_2}e^{\imath\omega\Omega(N)t_1}f_{N+1,N-1}, & \text{if } \begin{cases} n = N+1, \\ m = N-1, \end{cases} \\[2ex]
e^{-\imath\omega\Omega(N)t_2}e^{\imath\omega\Omega(N)t_1}f_{N+1,N+1}, & \text{if } \begin{cases} n = N+1, \\ m = N+1, \end{cases} \\[2ex]
e^{\imath\omega\Omega(N-1)t_2}e^{-\imath\omega\Omega(N-1)t_1}f_{N-1,N-1}, & \text{if } \begin{cases} n = N-1, \\ m = N-1, \end{cases} \\[2ex]
e^{-\imath\omega\Omega(N)t_2}e^{-\imath\omega\Omega(N-1)t_1}f_{N-1,N+1}, & \text{if } \begin{cases} n = N-1, \\ m = N+1, \end{cases}
\end{cases}
\end{aligned}
\tag{26}
$$

where we have defined

$$
\begin{aligned}
\Omega(N) &= \frac{1}{2}[(N+2)f^2(N+2) - Nf^2(N)], \\
f_{N+1,N-1} &= \sqrt{N(N+1)}f(N)f(N+1), \\
f_{N+1,N+1} &= -(N+1)f^2(N+1), \\
f_{N-1,N-1} &= -Nf^2(N), \\
f_{N-1,N+1} &= \sqrt{N(N+1)}f(N)f(N+1),
\end{aligned}
\tag{27}
$$

and all other matrix elements in (26) are identically zero. So the non zero transition probabilities are only $|N\rangle \rightarrow |N-1\rangle$ and $|N\rangle \rightarrow |N+1\rangle$. Let us find the decay amplitude

$$
\Gamma_{N\rightarrow N-1} = \mathrm{Tr}_S[|N-1\rangle\langle N-1|\hat{\rho}_S(t)]. \tag{28}
$$

From (25)–(28) we find

$$
\begin{aligned}
\Gamma_{N\rightarrow N-1} &= \langle N-1|\hat{\rho}_S(t)|N-1\rangle, \\
&= \frac{\omega}{2m\hbar}\int_0^t\int_0^t\int_{-\infty}^{\infty} dt_1 dt_2 dk\left[\frac{|f_1(\omega_k)|^2}{e^{\beta\hbar\omega_k}-1}\left(e^{\imath\omega_k(t_2-t_1)} + e^{\beta\hbar\omega_k - \imath\omega_k(t_2-t_1)}\right)\right. \\
&\qquad \left.\times\left(e^{\frac{\imath\omega}{2}[\gamma(N)-\gamma(N-1)](t_2-t_1)}Nf^2(N)\right)\right].
\end{aligned}
\tag{29}
$$

Using the following substitutions

$$
\begin{aligned}
u &= t_2 - t_1, \\
v &= t_2 + t_1, \\
du dv &= 2dt_2 dt_1,
\end{aligned}
\tag{30}
$$

the integrals over time can be evaluated simply and we find

$$\Gamma_{N\to N-1} = \frac{\omega t}{m\hbar}\int_{-\infty}^{\infty} dk \frac{|f_1(\omega_k)|^2}{e^{\beta\hbar\omega_k}-1} N f^2(N)\left[\frac{\sin[\omega_k+\omega\Omega(N-1)]t}{\omega_k+\omega\Omega(N-1)} + e^{\beta\hbar\omega_k}\frac{\sin[\omega_k-\omega\Omega(N-1)]t}{\omega_k-\omega\Omega(N-1)}\right]. \tag{31}$$

In large-time limit we have

$$\lim_{t\to\infty}\frac{\sin(\alpha t)}{\pi\alpha} = \delta(\alpha), \tag{32}$$

and by taking a linear dispersion relation $\omega_k = c|k|$, we finally find

$$\Gamma_{N\to N-1} = \frac{\omega t\pi}{m\hbar c}\left[N f^2(N)|f_1(\omega\Omega(N-1))|^2\right]\frac{e^{\beta\hbar\omega\Omega(N-1)}}{e^{\beta\hbar\omega\Omega(N-1)}-1}. \tag{33}$$

The absorbtion amplitude can be obtained similarly as

$$\Gamma_{N\to N+1} = \frac{\omega t\pi}{m\hbar c}\left[(N+1) f^2(N+1)|f_1(\omega\Omega(N))|^2\right]\frac{1}{e^{\beta\hbar\omega\Omega(N)}-1}. \tag{34}$$

For non deformed case $(f(N)\equiv 1, \Omega(N)\equiv 1)$, we find

$$\begin{aligned}\Gamma_{N\to N-1} &= \frac{\omega t\pi}{m\hbar c} N|f_1(\omega)|^2\frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega}-1},\\ \Gamma_{N\to N+1} &= \frac{\omega t\pi}{m\hbar c}(N+1)|f_1(\omega)|^2\frac{1}{e^{\beta\hbar\omega}-1}.\end{aligned} \tag{35}$$

When the reservoir is in its ground state that is $(\beta\to+\infty)$, then from (34) and (35) it is clear that there is only decay rates and energy flows from the oscillator to the reservoir.

3 Summary and Conclusion

In this paper we applied the minimal coupling method to a dissipative quantum harmonic oscillator. The reduced density matrix and the transition probabilities obtained. The results were compatible with the non deformed case.

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